Research Topic:

Investigation into the surface area of a traditional Chinese porcelain, through area of surface of revolution and optimization to minimize the packaging material used.

Research question:

What is the surface area of a porcelain, modelled through Lagrange interpolation, and measured through integration of the area of surface of revolution, so as to investigate the minimum surface area of the protection material through optimization?

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1.Introduction

My hometown, Fujian located in south eastern China, tops the country in the tea yield, output value and export growth rate, accounting for the gravel soil which is suitable for tea's growth. Thus, growing up in a local family in Fujian, I was passionate about intangible cultural heritage such as porcelains and tea culture. However, as people's demand for porcelain and glassware increases, more protection materials for those fragile products are needed for delivery, resulting in plastic pollution.

Figure 1 A picture of my porcelain used for storing tea.

Hence, I would like to investigate the surface area of this porcelain and compare how much material is wasted. Although it is vital to promote Chinese cultural heritage, people should also consider environmental-friendly policies to reduce plastic usage. When I looked at the porcelains, an idea emerged: What if I can make the use of plastic packages more efficiently to reduce waste? I began my investigation by exploring the surface area of these porcelains and looking into the difference in area with its protective plastic materials. According to online information, the area of the surface of revolution by integration can be implemented to calculate this porcelain through rotation. As the porcelains were always fragile during delivery, this investigation played an essential part in reducing resource waste and also promoting Chinese culture.

2. Aim

I have learnt optimization and noticed the volume of revolution in the IB syllabus. Moreover, given that the formula for the volume of a sphere is $\frac{4}{3}\pi r^3$and then the surface area of a sphere is $4\pi r^2$. Therefore, I wonder if there are any possibilities to calculate the surface area using a similar formula from volume of revolution. The formula of volume of revolution in the interval of $[a, b]$ to calculate the volume is given in the IB syllabus:

$$V = \pi \int_a^b [f(x)]^2 dx, x \in [a, b] \ \textbf{(1)}$$

Accordingly, I found out that the area of surface of revolution can be calculated through the below formula in the interval of [a, b]:

$$S = 2\pi \int_a^b f(x)\, ds \ \ \textbf{(2)}$$

I realised this formula can be applied to my porcelain, so the arc length of this porcelain can be simulated on a grid/coordinate. Moreover, the use of a piecewise function and Lagrange interpolation can be extended to find the surface area.

3.Background information

In order to determine the surface area, the formula of area of surface of revolution (integration) is derived from the concept of area below the curve or integration (the proof is shown in 3.2. section). Through rotating the piecewise function of the graph which is determined by the coordinates using knowledge of Lagrange interpolation, the 3D model of the porcelain can be constructed, and its surface area will be determined. Furthermore, the concept of optimization and trigonometry will be utilised to calculate the minimum surface area, and investigate the efficiency in allocating the resources in packaging materials.

3.1. Methodology:

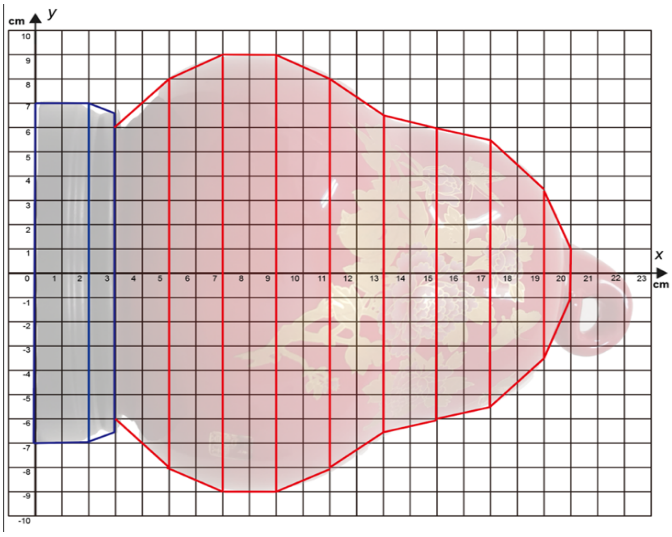
1. Brainstorm to look for the approaches of calculating the surface area of the porcelain, then investigate the formula of area of surface revolution through fitting polynomials to a set of points (See the below section).

2. Take a photo of the plastic materials and calculate its surface area by multiplying its width and length.
3. Take a photo of the porcelain from the perspective of orthographic projection (top and front).
4. Insert the orthographic projection into google drawing and Photoshop, and plot the picture on a graphic grid to determine its shape.
5. Use Lagrange interpolating polynomials to determine the functions of the ordinated orders on the graph.
6. Use Desmos to present the piecewise function depending on the determined orders.
7. Use GeoGebra to construct a 3D model of the porcelain.
8. Calculate the surface area of the Chinese porcelain and the usage efficiency of the packaging materials through GDC (Graphic display calculator).
9. Employ optimization to minimize the amount of plastic materials used to package the whole porcelain.
10. Assess the strengths and weaknesses of the exploration, then state the implication of the final results.

3.2. Exploration of the formula of the area of surface revolution:

The concept of integration and calculus is used to determine the formula, and it can be processed through briefly splitting irregular shapes of the porcelain into regular frustums. The magnification of actual porcelain and image sketched by Google drawing and Photoshop is 1 (image size/actual size).

Figure 2. A picture of the porcelain divided into 11 pieces at the ratio of 1:1.



In Figure. 2, it is obvious that the porcelain, which is symmetrical with the x -axis, can be divided into 11 frustums lying along the x -axis. The bottom line of this porcelain also fits with the y -axis. As the porcelain is almost a symmetrical pattern, it's possible to be formed through the frustum rotation around the x -axis. In Figure. 3, to determine the surface area of the porcelain, the surface area of each frustum of the cone (without its top) is calculated through similarity of the cones (concept from similar triangle). In the narrow and wide truncated cones respectively, the radii of the inclined surface area are R_1 and R_2, and the length of slants are s_1 and (s_2 - s_1). Therefore, it's clear to get the equation between the heights and radii of the frustum of the cone:

$$\frac{R_1}{R_2} = \frac{s_2 - s_1}{s_2}.$$

Through rearranging the formula to solve S2, I can get the following equation **(3):**

$$\frac{R_1}{R_2} = \frac{s_2 - s_1}{s_2}$$
$$R_1 s_2 = R_2(s_2 - s_1)$$
$$R_1 s_2 = R_2\, s_2 - R_2\, s_1$$
$$s_2(R_1 - R_2) = -R_2\, s_1$$
$$s_2 = \frac{R_2\, s_1}{R_2 - R_1}\ \textit{(3)}$$

In the view of knowledge of surface area of cone is given by $S = 2\pi r l$, the lateral surface area of the frustum is:

$$S = (Lateral\ SA\ of\ large\ cone) - (Lateral\ SA\ of\ smaller\ one)$$

$$= \pi R_2 s_2 - \pi R_1(s_2 - s_1)\ \textbf{(4)}$$

Expand and simplify the previous equation by substitute equation **(3)** to the equation **(4)**, then I can get the equation **(5)**:

$$= \pi R_2\left(\frac{R_2 s_1}{R_2 - R_1}\right) - \pi R_1\left(\frac{R_2 s_1 - s_1(R_2 - R_1)}{R_2 - R_1}\right)$$

$$= \frac{\pi R_2^2 s_1}{R_2 - R_1} - \frac{\pi R_2 R_1 s_1}{R_2 - R_1} + \frac{\pi R_1 s_1 (R_2 - R_1)}{(R_2 - R_1)}$$

$$= \frac{\pi R_2^2 s_1 - \pi R_2 R_1 s_1 + \pi R_1 s_1 (R_2 - R_1)}{R_2 - R_1}\ \textbf{(5)}$$

$(R_2 - R_1)$ can be cancelled out the denominator, so I can get the result of SA, shown in equation **(6):**

$$= \pi s_1 \frac{R_2^2 - R_1^2}{R_2 - R_1} = \pi s_1 \frac{(R_2 + R_1)(R_2 - R_1)}{R_2 - R_1}$$

$$= \pi s_1 \frac{R_2^2 - R_2 R_1 + R_1 R_2 - R_1^2}{R_2 - R_1}$$

$$= \pi s_1 (R_2 + R_1)\ \textbf{(6)}$$

As that the slants(s) of the frustum is considered as the slope in the graph (Fig.2), I replace s_1 with ds to get the final equation: $S = \pi s_1(R_2 + R_1)ds$

$$= \pi s_1\big(f(x_{i-1}) + f(x_i)\big)ds \quad \textbf{(7)}$$

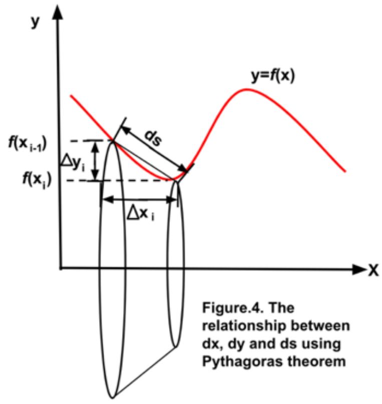


Figure.4. The relationship between dx, dy and ds using Pythagoras theorem

Where $ds = \sqrt{\Delta x_i^2 + \Delta y_i^2} = dx_i\sqrt{1 + (\frac{\Delta y_i}{\Delta x_i})^2}$

(according to Pythagoras theorem, see Figure.4).

Moving on, through the mean value theorem and randomly selecting a point $x_i^* \in [x_{i-1}, x_i]$ such that $f'(x_{i'}) = \frac{\Delta y_i}{\Delta x} = \frac{dy_i}{dx}$, where $x_{i'}$ represents a point different from x_i. The derivative ds can be written as $dx\sqrt{1 + \left(f'(x_i^*)\right)^2}$.

Additionally, since $\boldsymbol{f(x)}$ is continuous, by the Intermediate Value theorem, there is a point $x_{3i} \in [x_{i-1}, x_i]$, such that $f(x_i^*) = \frac{1}{2}\big(f(x_{i-1}) + f(x_i)\big)$. Therefore, I can get:

$$S = 2\pi f(x_i^*)\sqrt{1 + \left(f'(x_i^*)\right)^2}\, d\boldsymbol{x} \quad \textbf{(8)}$$

As a result, the approximate surface area of the whole surface of revolution is given by:

$$Surface\ Area \approx \sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1 + (f'(x_i))^2}dx \quad \textbf{(9)}$$

As $x_i^*$and $x_i^{**}$are evaluated from the interval $[x_{i-1}, x_i]$, the continuity and infinity of the function allow us to take the limit as $n \to \infty$. Through this approach, the limit works the same as Riemann sum even with the two different evaluation points. Thus, the result is:

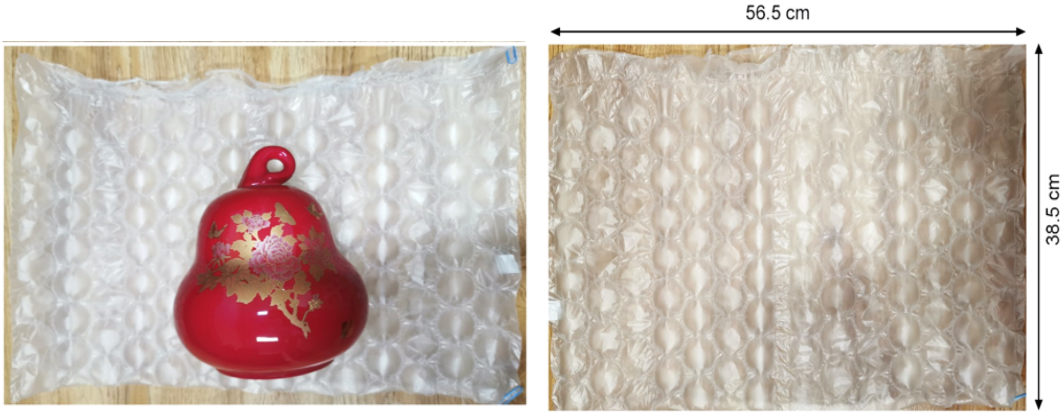
$$Surface\ Area = \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1 + (f'(x_i))^2}dx$$

$$= \int_a^b 2\pi f(x_i)\sqrt{1 + (f'(x_i))^2}dx$$

$$= \int_a^b 2\pi f(x)\sqrt{1 + (\tfrac{dy}{dx})^2}dx \quad \textbf{(10)}$$

3.3. Measurement of the surface area of the plastic materials (protective materials for the delivery), which can perfectly fit the porcelain.

Figure 5: A picture of the plastic materials (width and length are shown).



In order to calculate the area of surface of the plastic materials (protective materials), the width and length are measured by tape rulers which are 38.5 cm and 56.5 cm respectively. Then the surface area of the materials will be measured through the equation **(11)**:

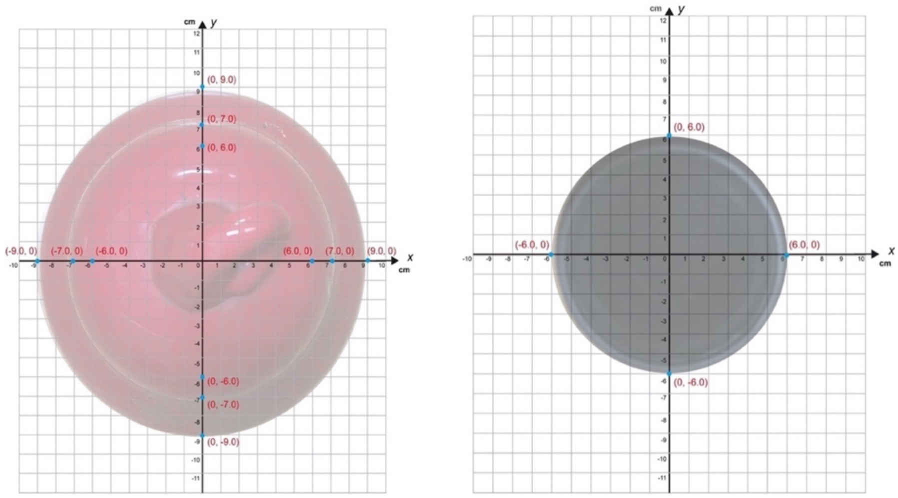
$$Surface\ area\ of\ rectangle\ = width \times length$$

$$= 38.5 \times 56.5\ = 2175.25\ cm^2\ \textbf{(11)}$$

However, the manual process of measurement has human errors that will hinder the accuracy of final results.

3.4. Measurement of porcelains (through perspectives of orthographic projection: top, bottom and front) and plotting coordinates of the porcelains.

Figure 6: A photo of the Chinese porcelain from the perspective of its top.



Due to its circular shape, in order to further determine the surface area of the Chinese porcelain, the width of the Chinese porcelain is required. Therefore, the method of taking a photo of a porcelain is adopted, through orthographic projection: top, front and bottom, is adopted. Considering the difficulty of measuring the radius directly, the choice of measuring the circumference through tape measure can be made, and then value of radii can be derived through the formula **(12):**

$$r = \frac{Circumference}{2\pi} \quad \textbf{(12)}$$

Where the result can also be shown in Table.1.

Table.1

Order	Circumference (cm)	Radii(cm)
1	**37.7**	**6.0**
2	**43.9**	**7.0**
3	**56.6**	**9.0**
4	**37.7**	**6.0**

3.5. Insert the orthographic projection into google drawing and Photoshop, and plot the picture on a graphic grid to determine its shape.

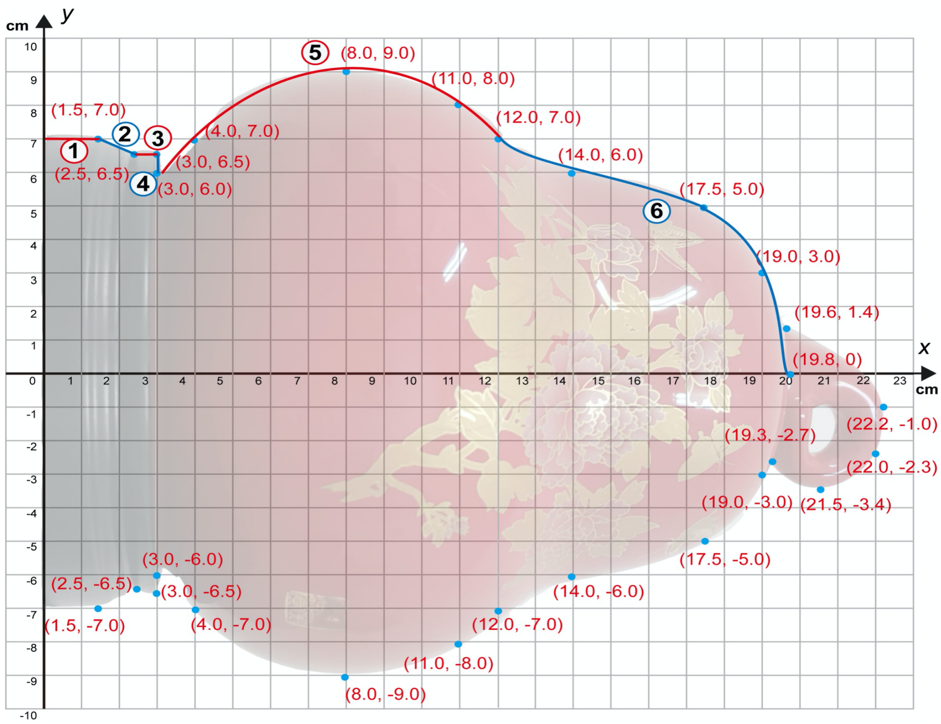
Here I would like to guess the equations depending on the points given in Figure.7. Initially, I divide the data points into two groups which can be shown in Table.2 and Table.3, up to their function property.

Data points in Table.2 are shown in the below, whose x and y in the coordinates are expressed.

Table. 2. Points in the first 4 frustums of porcelain, which is divided into 6 frustums.

the 1st frustum		the 2nd frustum		the 3rd frustum		the 4th frustum	
x	y	x	y	x	y	x	y
0	7	2.5	6.5	1.5	2.5	3	6
1.5	7	3	6.5	2.5	6.5	3	6.5

Figure 7: A photo of the Chinese porcelain from the perspective of its right.



In Figure. 7, the shape of the whole porcelain can be split into 6 different frustums, whose outlines are shown by different colours of lines or curves. Specifically, it is apparent that the first four lines are presented as a form of linear functions: $y = kx + b$, which can be determined by the undetermined coefficient method and shown by the below function group:

$$\begin{cases} f1(x): y = 7, x \in [0,1.5] & \mathbf{(13)} \\ f2(x): y = -0.5x + 7.75, x \in [1.5,2.5] & \mathbf{(14)} \\ f3(x): y = 6.5, x \in [2.5,3] & \mathbf{(15)} \\ f4(x): x = 3, y \in [6,6.5] & \mathbf{(16)} \end{cases}$$

However, when I rotate the function along the $\boldsymbol{x}$ -axis, the vertical line *f4*: $\boldsymbol{x} = 3$ won't appear in the final result. Thus, it won't be considered in the final function group and illustrated in the final graph.

3.6. Use Lagrange interpolating polynomials to determine the functions of the ordinated orders on the graph.

Lagrange theorem is the method for determining a polynomials, which takes on specific values at random species.

As I can't determine the equation of the 5th and 6th frustum of a function, Lagrange's theorem is required to work out their functions based on the coordinates given in Table 3.

Table. 3 Coordinates in the 5th and 6th frustum of the porcelain.

Coordinates in the 5th frustum		Coordinates in the 6th frustum	
$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{x}$	$\boldsymbol{y}$
3.0	6.0	12.0	7.0
4.0	7.0	14.0	6.0
8.9	9.0	17.5	5.0
11.0	8.0	19.0	3.0
12.0	7.0	19.8	0.0

Given a set of n points on a graph, any possible polynomials of sufficiently high degree that go through all n of the points. There is, however, just one polynomial of degree less than n that will go through them all. Firstly, Let's suppose that I have a set of n points:

$$(\boldsymbol{x}_1, \boldsymbol{y}_1), (\boldsymbol{x}_2, \boldsymbol{y}_2), (\boldsymbol{x}_3, \boldsymbol{y}_3), (\boldsymbol{x}_4, \boldsymbol{y}_4), (\boldsymbol{x}_5, \boldsymbol{y}_5), (\boldsymbol{x}_6, \boldsymbol{y}_6) \ldots\ldots (\boldsymbol{x}_n, \boldsymbol{y}_n)$$

Let's assume that a polynomial of degree n-1 to them is fitted, which can be shown when the number of function is n:

$$y = \sum_{i=1}^{n} y_i L_i(x) \quad \textbf{(17)}$$

where $L_i(x)$, i=1, n are n Lagrange polynomials, which are polynomials of degree n-1 defined by:

$$L_i(x) = \prod_{j=1, j\neq i}^{n} \frac{x-x_j}{x_i-x_j} = \frac{(x-x_1)(x-x_2)\ldots(x-x_i)(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_1)(x_i-x_2)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_1-x_n)} \quad \textbf{(18)}$$

At first encounter, the above coordinates of the fifth frustums will be considered, which are (3.0, 6.0), (4.0, 7.0), (8.9, 9.0), (11.0, 8.0). Writing more explicitly, the first four Lagrange polynomials can be shown through the following expansion:

$$L_1(x) = \frac{(x-4)(x-8.9)(x-11)(x-12)}{(3-4)(3-8.9)(3-11)(3-12)}$$

$$= \frac{x^4 - 35.9\,x^3 + 464.3\,x^2 - 2521.6x + 4699.2}{-424.799}$$

$$\approx -0.0024\,x^4 + 0.0845\,x^3 - 1.093\,x^2 + 5.936\,x - 11.062$$

$$L_2(x) = \frac{(x-3)(x-8.9)(x-11)(x-12)}{(4-3)(4-8.9)(4-11)(4-12)} \approx -0.004\,x^4 + 0.127\,x^3 - 1.576\ \,x^2 + 7.962\,x - 12.844$$

$$L_2(x) = \frac{(x-3)(x-4)(x-11)(x-12)}{(8.9-3)(8.9-4)(8.9-11)(8.9-12)} \approx 0.005\,x^4 - 0.159\,x^3 + 1.621\,x^2 - 6.376\,x + 8.416$$

$$L_3(x) = \frac{(x-3)(x-4)(x-8.9)(x-12)}{(11-3)(11-4)(11-8.9)(11-12)} \approx -0.009\,x^4 + 0.237\,x^3 - 2.254\,x^2 + 8.489\,x - 10.898$$

$$L_2(x) = \frac{(x-3)(x-4)(x-8.9)(x-11)}{(12-3)(12-4)(12-8.9)(12-11)} \approx 0.004\,x^4 - 0.121\,x^3 + 1.117\,x^2 - 4.140\,x + 5.263、$$

According to the example of equation (11), I can get the final equation by using the equation: $f(x) = f(x_1)L_1(x) + f(x_2)L_2(x) + f(x_3)L_3(x) + f(x_4)L_4(x) + f(x_5)L_5(x)$. It gives the final result of the function of the fifth frustums:

$$f5(x){:}\,-0.0002x^4 + 0.003x^3 - \ 0.106x^2 + 1.762x + 1.873,\ \ x \in [3{,}12\,] \quad \textbf{(19)}$$

As soon as I got the fifth function, the final equation of the sixth as well as the last frustums can be given through the formula equation:

$$f6(x){:}\,y = -0.014x^4 + 0.842x^3 - 18.77x^2 + 183.88x - 660.543, x \in [12{,}19.8] \quad \textbf{(20)}$$

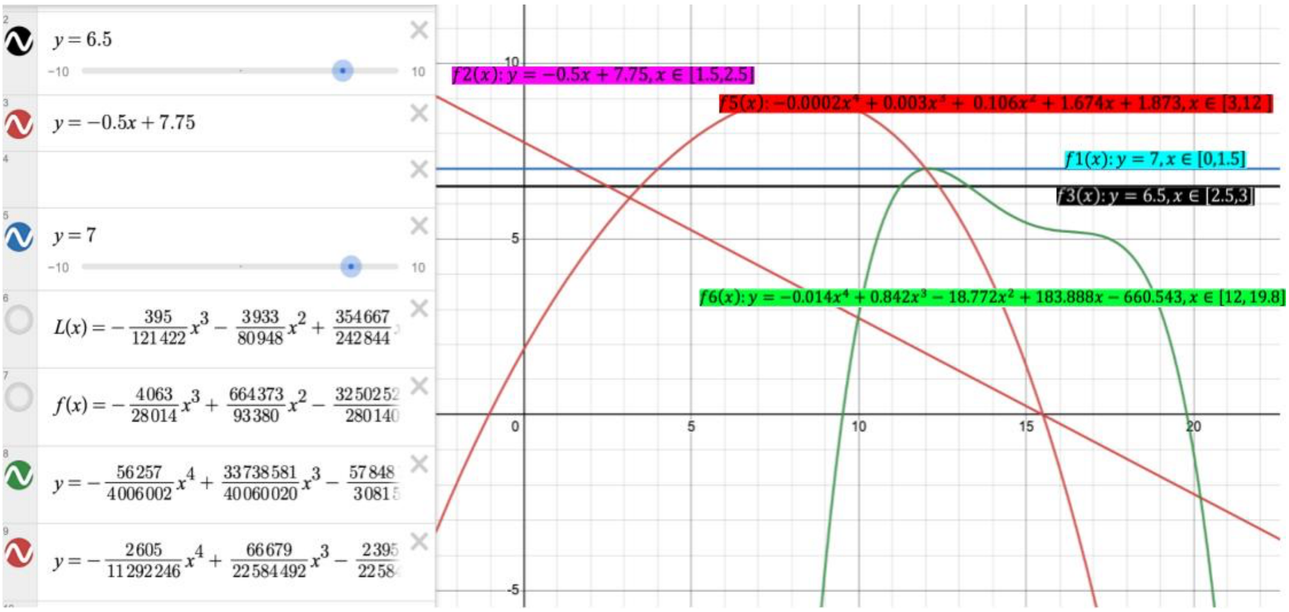
All functions are founded through Lagrange theorem and fundamental knowledge of undetermined coefficient methods, the inaccuracy still exist as the porcelain is not completely symmetrical.

After rearrangement and adjustment, $f4(\boldsymbol{x})$ as it won't appear in further rotation because it is vertical line that will still be the same after rotation, so the final function group will be:

$$\begin{cases} f1(x): y = 7, x \in [0,1.5] \\ f2(x): y = -0.5x + 7.75, x \in [1.5,2.5] \\ f3(x): y = 6.5, x \in [2.5,3] \\ f5(x): -0.0002x^4 + 0.003x^3 - \ 0.106x^2 + 1.762x + 1.873, x \in [3,12\ ] \\ f6(x): y = -0.014x^4 + 0.842x^3 - 18.772x^2 + 183.888x - 660.543, x \in [12, 19.8] \end{cases}$$

3.7. Use Desmos to present the piecewise function depending on the determined orders.

Figure 8.



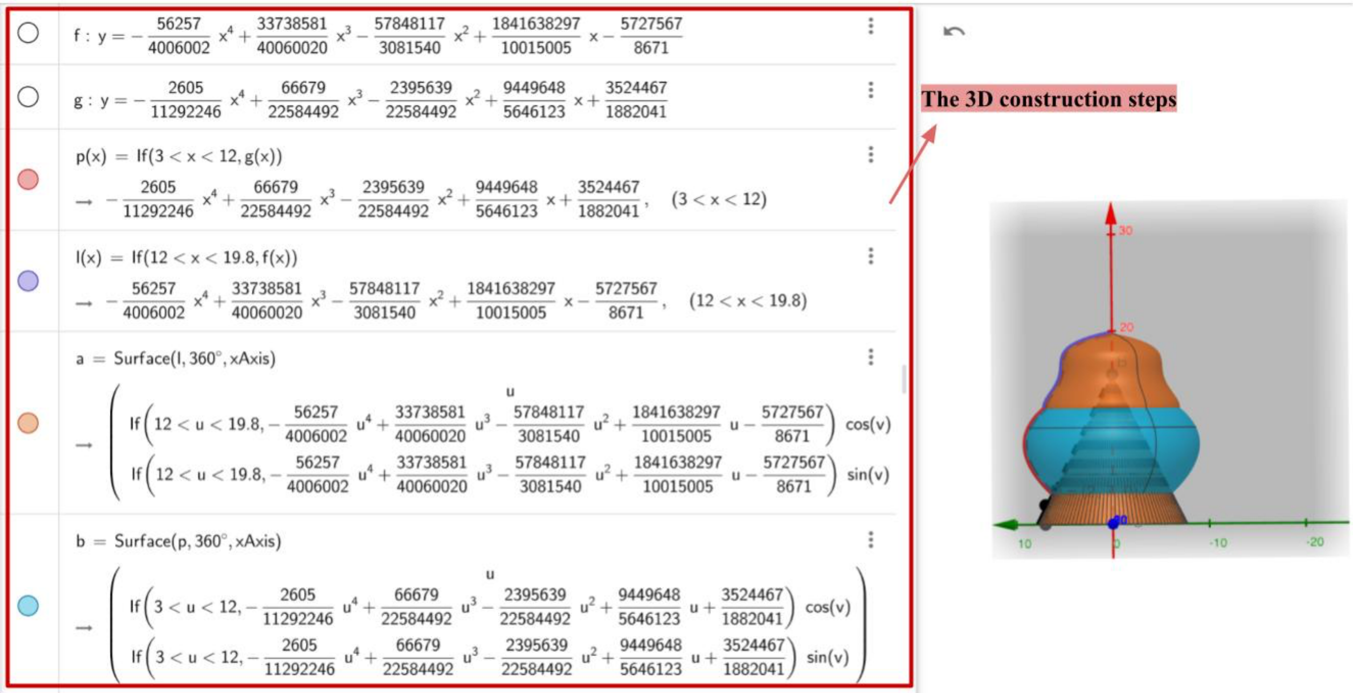
Moving on, I got the brief graph shape of Chinese porcelain for further rotation by inserting function groups on Desmos (Different functions are expressed by line in different colours), as the technology will help me sketch the functions more accurately and reliable compared to hand drawing. I used different colours and labels to distinguish them. However, the problems I would like to points out is the missing irregular object on the top of the porcelain, which can't be expressed by a continuous function (So I would discuss it later in Evaluation).

3.8. Use GeoGebra to construct a 3D model of the porcelain with the given functions

Initially, I entered the functions in the function group found by Lagrange's theorem to GeoGebra 3D calculator, notably, the domains are required to be established.

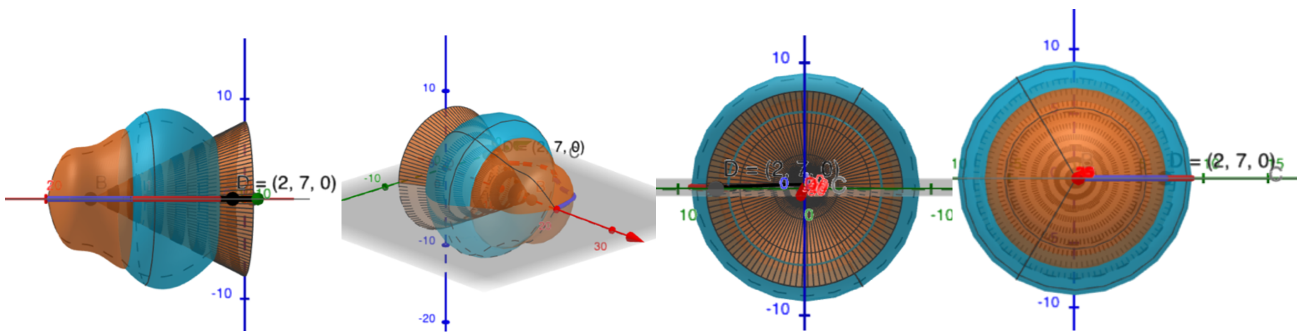
Later, the key step of constructing a 3D model in my investigation is to rotating the functions about the x -axis through entering the angle of revolution, which is 360° and written as a form of "Surface (function, 360°, x -Axis)" (See Figure.9).

Figure 9



By applying the procedures on other functions, I construct the 3D model of the Chinese porcelain, and Figure.10 -11 are the 3D models shown from different perspectives.

Figure 10-11: the construction of the 3D model of the porcelain.



3.9. Calculate the surface area of the porcelain.

3.9.1. Area of surface of revolution

As I have already known the formula for area of surface of revolution and the functions of best fit, I am able to calculate the surface area of the second, fourth and the last frustum of the porcelain. In the following calculation session, I will label the surface area through adding the number corresponding to its frustum. For example, the second frustum corresponds to "S_2". Besides, I will first calculate the surface area of the irregular shape and calculate the cylinder surface area later on. To begin, I know the following formula (10) from the exploration of theory:

$$S = \int_a^b 2\pi f(x)\sqrt{1 + (\frac{dy}{dx})^2}dx \quad \textbf{(10)}$$

With this formula, I will first calculates the surface area of second, fifth and sixth frustums of the porcelain. To begin with, the surface area of the second frustum of the porcelain is calculated through its function: $\boldsymbol{f}2(\boldsymbol{x})$: $\mathrm{y} = -0.5\boldsymbol{x} + 7.75, \boldsymbol{x} \in [1.5, 2.5]$, as it shows:

$$S_2 = 2\pi \int_{1.5}^{2.5} (-0.5x + 7.75)\sqrt{1 + (-0.5)^2}\, dx$$

$$= 2\pi \int_{1.5}^{2.5} (-0.5x + 7.75)\sqrt{3.5}\, dx$$

Hence, it can be simplified to $= 0.4\pi \times \sqrt{3.5} \cdot [-x^2 + 15.5x] \begin{matrix} 2.5 \\ 1.5 \end{matrix}$

The final answer is given by GDC (Graphic display calculator): $S_1 \approx 79.345\ (cm^2)$ **(21)**

Using the same formula, the surface area of other frustums can be determined through repeating steps. As I know the surface area of the third frustum, what I can get through its original function $f5(x)$: $-0.0002x^4 + 0.003x^3 - 0.106x^2 + 1.762x + 1.873, x \in [3, 12]$ is:

$$S_5 = 2\pi \int_3^{12} 0.0002x^4 + 0.003x^3 - 0.106x^2 + 1.762x + 1.873\ dx$$

As a result, I can get the final surface area through integration:

$$= 2\pi[-0.00004x^5 + 0.00075x^4 - 0.035333x^3 + 0.881x^2 + 1.873x] \begin{matrix} 12 \\ 3 \end{matrix}$$

$$= 101.12481 \times 2\pi \approx 635.386\ (cm^2)\ \textbf{(22)}$$

The surface area of the fifth frustum can be calculated through its original function $f6(x)$: $y = -0.014x^4 + 0.842x^3 - 18.772x^2 + 183.888x - 660.543, x \in [12, 19.8]$

$$S_6 = 2\pi \int_{12}^{19.8} -0.014x^4 + 0.842x^3 - 18.772x^2 + 183.888x - 660.543\ dx$$

Therefore, I can get its surface area: $S_5 \approx 365.408\ (cm^2)$ **(23)**

Although inaccuracy still exists due to approximation and systematic errors while measuring, the surface area of the second, fifth and sixth frustums are briefly obtained.

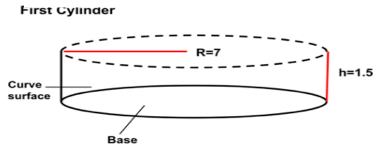
3.9.2. Surface area of cylinder

However, considering the cylinder shapes of the first and third frustum, two functions which are which are $y = 7, x \in [0,1.5]$ and $y = 6.5, x \in [2.5,3]$, are then considered to measure the surface area, as I have already known their radii and height respectively. Therefore, I can calculate their surface area using the formula of cylinder:

$$Surface\ area_{\ Cylinder} = 2\pi rh + 2\pi r^2$$

For the first cylinder (the first frustum) lying along the $\boldsymbol{x}$ -axis, I only need to calculate the area of its base and curved surface, so the calculation will be:

Figure12

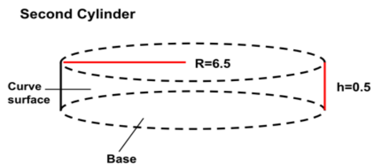


$$S_1 = 2\pi rh + \pi r^2$$

$$S_1 = 14\pi + 49\pi \approx 197.92\ (cm^2)\ \textbf{(24)}$$

For the second cylinder (the third frustum) lying along the $\boldsymbol{x}$ -axis, I have to calculate the surface area of the curved surface without the base because its base is excluded in the middle part of the porcelain, so the calculation is shown in the following steps:

Figure 13



$$S_3 = 2\pi rh$$

$$S_3 = 6.5\pi \approx 20.42\ (cm^2)\ \textbf{(25)}$$

3.9.3. Total surface area of the porcelain, usage of efficiency and percentage of waste

To get the total surface area of the Chinese porcelain, all the surface areas of the frustums gotten from either surface area of revolution or cylinder surface area formula need to be summed up to obtain the sum:

$$S_{sum} = S_1 + S_2 + S_3 + S_5 + S_6$$

Through calculation shown by step **(26),** the total surface area of the porcelain is obtained, but the results might not be as expected as the actual one because the porcelain is not completely symmetrical.

$$S_{sum} = 79.345 + 635.386 + 365.408 + 197.92 + 20.42 \approx 1298.479\ (cm^2)\ \textbf{(26)}$$

As I know the total surface area of the Chinese porcelain, I can calculate the usage efficiency of packaging materials through the formula:

$$Usage\ efficiency = \frac{Surface\ area_{porcelain}}{Surface\ area_{package\ materials}} = \frac{1298.479}{2175.25} \times 100\% \approx 59.693\%\ \textbf{(27)}$$

59.693% of usage of efficiency apparently suggests that there is big waste in package materials while packaging the porcelain, and the percentage can also be calculated based on the usage of efficiency through either ($1 - usage\ of\ efficiency$) or (diving difference in surface area of porcelain and package materials by the surface area of package materials) , as the following formula shows:

$$Percentage\ of\ waste = \frac{Surface\ area_{difference}}{Surface\ area_{package\ materials}} = \frac{2175.25-1891.465}{2175.25} = \frac{876.771}{2175.25} \approx 40.307\% \quad \textbf{(28)}$$

4. Find out the optimal surface area to package the porcelain.

From the above calculation of usage efficiency and percentage of waste, I wondered if there is more efficient way to lower the waste of materials like changing the shape of porcelain to the fit the packages. Back to the knowledge of optimization I learnt from IB math class, I would like to apply optimization to minimize the surface area of plastic package given a same size.

In order to determine the possibly smallest surface area of plastic materials of the same volume, I assume from Figure. 5 that the majority of plastic materials is used to package the fifth frustum whose body has the maximum radius in the whole porcelain (Figure. 7). Consequently, it is acceptable to view the whole porcelain as a cylinder object.

With reference the rule: $\frac{d}{dx}x^n = nx^{n-1}$, I can differentiate $f5(\boldsymbol{x})$ and find out the maximum radius through solving for $\boldsymbol{r}'(\boldsymbol{x}) = 0$.

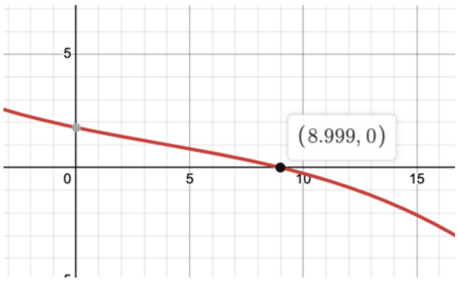
$$\frac{d}{dr}(-0.0002x^4 + 0.003x^3 - 0.106x^2 + 1.762x + 1.873\ )$$

$$= -0.0008\,x^3 + 0.009\,x^2 - 0.212\,x + 1.762$$

Now according to the first derivative equation, I can get the $\boldsymbol{r}$ through finding the value of $\boldsymbol{x}$ in the following equation first:

$$-0.0008\,x^3 + 0.009\,x^2 - 0.212\,x + 1.762 = 0 \quad \textbf{(29)}$$

Figure 14



By presenting the equation on Desmos, the function has only one root of which is 8.999, corrected to 4 significant figures. to ensure higher level of precision.

In the next step, I substitute 25.117 into the original function to find the maximum radius of the Chinese porcelain which is shown in the following equation, and the value of $\boldsymbol{f(x)}$ is the same as r:

$$f(x) = -0.0002(8.999)^4 + 0.003(8.999)^3 - 0.106(8.999)^2 + 1.762(8.999) + 1.873$$

$$\approx 10.02 \text{ (cm)}$$

As soon as I got the maximum radius of the porcelain, I could calculate the minimum surface area of plastic packaging materials required through employing the concept of optimization behind a cylinder based on the surface area and volume formula of a cylinder:

$$\text{Cylinder} = \begin{cases} SA\ (Suface\ area) = 2\pi rh + 2\pi r^2 h & (\mathbf{30}) \\ V(Volume) = \pi r^2 h & (\mathbf{31}) \end{cases}$$

To determine the minimum surface area, the relationship between known variables, height and radii, has to be determined, which are shown with respect to r. Firstly, divide the both side of the equation (30) by 2π, and then get the following equation:

$$hr + h = \frac{SA}{2\pi r^2}$$

Extract $(1 + r)$ on the left side of the equation, then divide both sides of the equation by $(1 + r)$. Finally, the relationship between surface area and height can be shown by the following step **(32):**

$$h = \frac{SA}{2\pi r^2(1+r)},\ SA = h(2\pi r^2(1 + r))$$

As for the relationship between height and volume, both sides of the equation (31) is divided by (πr^2), whose final result is shown in the below **(33)**:

$$h = \frac{V}{\pi r^2}\ (\mathbf{33})$$

As we have already known the relationship between height and its volume, $h = \frac{V}{\pi r^2}$, we can insert it back to the equation **(32),** and then differentiate it which are shown in the following step **(34):**

$$\frac{dSA}{dr}\left(h\left(2\pi r^2(1 + r)\right)\right) = \frac{dSA}{dr}\left(\frac{V}{\pi r^2} \times 2\pi r^2(1 + r)\right) = 4\pi r - \frac{2V}{r^2}\ \ \mathbf{(34)}$$

In order to determine the minimum surface area, the critical points have to be found by making the derivative equal to 0. Thus, the final equation (34) are supposed to be 0, given by the condition:

$$\because 4\pi r - \frac{2V}{r^2} = 0, \therefore V = 2\pi r^3\ (\mathbf{35})$$

When $V = 2\pi r^3$ $(\mathbf{35})$ is expressed in the term of maximum radii, the further verification **(36)** will be made based on the second derivative test, in order to verify if the surface area is the minimum. If the surface area is the minimum, the second derivative value given will be greater than 0, vice versa.

$$\frac{dSA}{dr}\left(4\pi r - \frac{2V}{r^2}\right) = 4\pi + \frac{2(2\pi r^3)}{r^3} = 4\pi + 4\pi = 8\pi, 8\pi > 0 \ \ (\mathbf{36})$$

Based on the second derivative test, it is evident that when $V = 2\pi r^3$, there is a minimum surface area of the packaging materials used. At the same time, compared with $V = \pi r^2 h$ shown in the equation **(31)**, it is obvious that

$$h = 2r = d \ \mathbf{(37)}$$

Consequently, the minimum of surface area is achieved when the height is equals to $2r$ as well as the diameter, which can be related back to the formula of the surface area of cylinder, where $r = 10.02$:

$$SA \ = 2\pi rh = 2\pi r(2r) = 4\pi r^2 \approx 1261.6686 \ cm^2 \ (\mathbf{38})$$

$$1261.6686 \ \text{cm}^2 < 2175.25 \ \text{cm}^2 \ (original\ SA\ of\ package\ materials)$$

Which is the optimal surface area of package materials used to packaging this porcelain. The surface area of the new porcelain is much less than the original surface area of the package materials, which is $2175.25 \ \text{cm}^2$. Even though optimization to minimize the surface area in an assumed same volume, the porcelain has already been manufactured whose shape is hard to change. As a result, this method cannot allow the porcelain to be completely covered.

$$\frac{2175.25 \ \text{cm}^2}{1261.6686 \ \text{cm}^2} \approx 1.7241$$

The above calculation shows that the previous surface of package materials used is divided by the new one, which can be noticed that the amount of waste can be reused to package 1.724 porcelains of the same size. Furthermore, the investigation of optimization reveals the underlying inefficient allocation of resources when manufacturing those porcelain.

5.Evaluation:

5.1. Strength:

1. I used the extended knowledge to resolve real-world issues such as environmental degradation and pollution through minimizing the package waste depending on the surface area.
2. I increased the precision to more than 3 places during my calculation and selected 12 coordinate orders in the graph for the application of Lagrange's theorem to increase the data accuracy in calculation.
3. I used mathematical methods to calculate the circumference (find the radius and then use the formula of circumference of the circle) instead of manually measuring it.

5.2. Weakness / limitation and corresponding improvement:

1. The irregular part of the porcelain is not measured, resulting in a smaller actual surface area compared with the theoretical one. The surface area of the irregular part can be calculated by seeing the irregular part as a sphere or using other non-mathematical methods. For example, tying the object tightly with scotch tape and colour it with a black marker. Next, take off the tape, expand it, which is presented as an irregular black object, and then photograph it. Later, the photo can be uploaded to Desmos to determine the function of its frustums' outline, and then calculates their surface area by using the same concept of area below the curve or integration.
2. Human error and approximation are the major and potential issue resulting in data inaccuracy, so it should be taken into account (parallax). Additionally, Optimization can be more specific by splitting the porcelain into even smaller sections to ensure accuracy and data reliability. I can improve it through establishing more data points in the graph to determine the function and increase the accuracy through Lagrange's theorem. Furthermore, it is also acceptable to increase the precision to 6 places in calculation to increase data accuracy and validity.

In conclusion, through the concept of Lagrange's theorem, which is derived from the similarity of triangles, to find the polynomial curve of the frustums. Even though there are some inaccuracies when measuring the porcelains' actual shape, the brief curve can be obtained through Lagrange theorem. The investigation reveals the underlying environmental problems due to the amount of waste when packaging the porcelain, whose surface area is around but requires about of the package materials. Through this investigation, if the manufacturer changes the shape of the porcelain of the same size through using the optimal maximum radii, one porcelain only needs about of package materials for protection during delivery. By doing so, it addresses the problem of negative externality of production due to the amount of waste that can be reused to protect 1.72 porcelain of the same size, and reduce the cost of production while promoting the environmental sustainability and resources.

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